Solutions

School of Mathematics and Statistics
Carleton University
Math. 1004A, Fall 2013

MOCK TEST 6

Any non-programmable calculator permitted, 1 blank sheet permitted for roughs

Print Name : ______________________

Student Number: ______________________

Tutorial Section (A1, A4, . . .): ______________________

PART I: Multiple Choice Questions

(Choose and CIRCLE only ONE answer - No part marks here.)

1. **[3 marks]** Evaluate $\int_0^\infty 3xe^{-x}\,dx$.

   (a) 3, (b) 0, (c) 1, (d) 4

2. **[3 marks]** Evaluate $\int_0^\infty x^2 3^{-x}\,dx$.

   (a) $\frac{2}{\ln 3}$, (b) 1, (c) $\frac{1}{(\ln 3)^2}$, (d) $\frac{2}{(\ln 3)^3}$

3. **[3 marks]** Evaluate $\int_2^4 \sqrt{x^2-4}\,dx$

   (a) $\sqrt{3}-\ln(2+\sqrt{3})$, (b) $4\sqrt{3}-2\ln(2+\sqrt{3})$, (c) $12\sqrt{3}$, (d) $4\sqrt{3}-\ln(2+\sqrt{3})$

4. **[3 marks]** Find the area enclosed by the curves $y=2x^2-5$ and $y=3$.

   (a) 8, (b) $\frac{1}{4}$, (c) $\frac{64}{3}$, (d) $\frac{2}{3}$,

5. **[3 marks]** Evaluate $\int_0^1 \sqrt{1-x^2}\,dx$

   (a) 1, (b) $\frac{\pi}{4}$, (c) $\frac{\pi}{2}$, (d) $\frac{\pi}{3}$,

PART II: Show all work here and give details.
No additional pages will be accepted

6. **[10+5 marks]**

   a) Find the volume of the solid of revolution obtained by rotating the region bounded by the lines $x=1$, $x=2$, $y=x$ and $y=-x$ about the y−axis.

   b) Find an expression for the solid of revolution obtained by rotating the region bounded by the lines $y=2x$, $y=3x$ and $x=1$ about the x−axis. DO NOT EVALUATE the constants nor the integral.

a) The points of intersection are: $(1,1)$, $(2,2)$, $(1,-1)$ and $(2,-2)$.

Use a vertical slice. Then $r_{out}=x$, $r_{in}=x-dx$

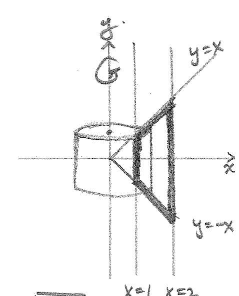
height $= x-(-x) = 2x$, and $1 \le x \le 2$.

$\therefore$ Slice volume when rotated about y-axis

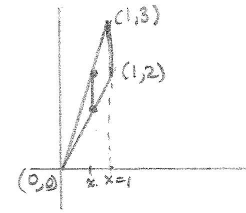
$= \pi(r_{out}^2 - r_{in}^2)(ht) = \pi\left(x^2-(x-dx)^2\right)2x$

$= 2\pi x\left(2x\,dx - dx^2\right)$

$\therefore$ Total Vol $= 4\pi\int_1^2 x^2\,dx = 4\pi \frac{x^3}{3}\Big|_1^2 = \frac{4\pi}{3}(8-1) = \boxed{\frac{28\pi}{3}}$



b) Pts of intersection are $(0,0), (1,2), (1,3)$.
Use a vertical slice. End pts are $(x, 2x)$ and $(x, 3x)$.



$\therefore r_{in} = 2x, \quad r_{out} = 3x, \quad ht = dx$

$\therefore$ slice vol $= \pi\left((3x)^2 - (2x)^2\right) dx = 5\pi x^2 dx$.

$\therefore$ Volume $= \int_0^1 5\pi x^2 dx$.

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1. Use Table Method to get $\int 3xe^{-x} dx = -3(xe^{-x} + e^{-x})$

$\therefore I = \int_0^\infty 3xe^{-x} dx = \lim_{T\to\infty} \int_0^T 3xe^{-x} dx = \lim_{T\to\infty} \left[-3(Te^{-T} + e^{-T}) + 3\right]$.

$= -3\lim_{T\to\infty}\left(\frac{T}{e^T}\right) - 3\lim_{T\to\infty} e^{-T} + 3 = 0 + 0 + 3 = \underline{\underline{3}}$

(where we used L'Hopital's Rule in 1^{st} limit).

2. Write $3^{-x} = e^{-x\ln 3}$ & use Table Method to get

$$\int_0^T x^2 3^{-x} dx = \left(-\frac{x^2 e^{-x\ln 3}}{\ln 3} - \frac{2xe^{-x\ln 3}}{(\ln 3)^2} - \frac{2e^{-x\ln 3}}{(\ln 3)^3}\right)\Bigg|_{x=0}^{x=T}$$

$$\lim_{T\to\infty} \int_0^T x^2 3^{-x} dx = (0 - 0 - 0) - \left(0 - 0 - \frac{2}{(\ln 3)^3}\right) = \frac{2}{(\ln 3)^3}.$$

(by L'Hopital's Rule).

3. Let $x = 2\sec\theta$, $dx = 2\sec\theta\tan\theta\, d\theta$, $\sqrt{x^2-4} = 2\tan\theta$

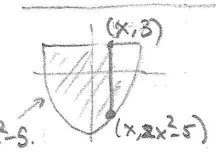
$\therefore I = \int_2^4 \sqrt{x^2-4}\, dx = 4\int_0^{\pi/3} \sec\theta\tan^2\theta\, d\theta$. (see and Example 391 / p403)

$= \left[2\tan\theta\sec\theta - 2\ln|\sec\theta + \tan\theta|\right]\Big|_0^{\pi/3}$

$= 4\sqrt{3} - 2\ln|2+\sqrt{3}|$

N.B. typo on p403 Ex. 391. Divide $x\sqrt{x^2-4}$ by 2

4.



Use Vertical slice (all pts as fns of x).

Area of a slice $= \left(3 - (2x^2-5)\right) dx$

$= (8 - 2x^2)\, dx$

$\therefore$ Area of region $= \int_{-2}^{2} (8 - 2x^2)\, dx = 64/3$.

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5. $x = \sin\theta$, $dx = \cos\theta\, d\theta$, $x = 0, \theta = 0$ & $x = 1, \theta = \pi/2$.

$\therefore I = \int_0^{\pi/2} \cos\theta \cdot \cos\theta\, d\theta = \int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta$

$= \frac{\pi}{4} + \frac{1}{2}\left[\frac{\sin 2\theta}{2}\right]\Big|_0^{\pi/2} = \pi/4$.
